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LETTER TO THE EDITOR

Proof of integrability for five-wave interactions in a case with unequal coupling constants

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Abstract. Complete integrability is proved for a system of ODE describing the double three-wave interaction between five waves, when the coupling constant in one triplet is twice as large as in the other. Known results from the Painlevé analysis for this case are combined with the Yoshida–Kovalevskaya approach to gain insight about the degree of the missing first integral. This integral of degree six is then obtained via a direct search based on irreducible forms, elementary building blocks for polynomial first integrals in involution with the Manley–Rowe invariants.

In this letter we study the system of coupled ODE describing the non-linear interaction between five waves coupled in two triplets, where the common wave is a pump wave. The equations for the slow evolution of the wave amplitudes are given in complex notation (see, e.g., Verheest (1987c)):

$$\dot{a}_0 = \mathrm{i}a_1a_2 + \mathrm{i}\mu a_3a_4 \qquad \dot{a}_1 = \mathrm{i}a_0\bar{a}_2 \qquad \dot{a}_2 = \mathrm{i}a_0\bar{a}_1$$
$$\dot{a}_3 = \mathrm{i}\mu a_0\bar{a}_4 \qquad \dot{a}_4 = \mathrm{i}\mu a_0\bar{a}_3 \tag{1}$$

plus their complex conjugates. This is a special case of the interaction between N triplets, as studied before in ocean wave dynamics and/or plasma turbulence by Watson *et al* (1976) and Menyuk *et al* (1983a, b). In the latter papers the Painlevé analysis was used to determine the integrability of systems such as (1), viewed as Hamiltonian, derivable from

$$H - a_0\bar{a}_1\bar{a}_2 + \bar{a}_0a_1a_2 + \mu(a_0\bar{a}_3\bar{a}_4 + \bar{a}_0a_3a_4) \tag{2}$$

with canonical equations in the form (Verheest 1987a):

$$\bar{a}_j = \mathrm{i}\partial H/\partial\bar{a}_j \qquad \dot{\bar{a}} = -\mathrm{i}\partial H/\partial a_j \qquad j = 0, \ldots, 4. \tag{3}$$

Thus complex conjugates are canonically conjugate Hamiltonian variables.

From the Painlevé analysis carried out by Menyuk *et al* (1983a) it followed that (1) was deemed integrable (here for simplicity leaving out the frequency detunings and taking $N = 2$) when either $\mu = 1$ or $\mu = 2$ or $\frac{1}{2}$ (depending on the order in which the waves are numbered). There are already four first integrals at arbitrary μ, namely three Manley–Rowe relations

$$E_A = a_1\bar{a}_1 - a_2\bar{a}_2 \qquad E_B = a_3\bar{a}_3 - a_4\bar{a}_4$$
$$E_0 = 2a_0\bar{a}_0 + a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3 + a_4\bar{a}_4 \tag{4}$$

besides H itself. As these four first integrals are independent and mutually in involution, a fifth such one is necessary to prove complete integrability (Pars 1965).

For the case where $\mu = 1$, this was done (for arbitrary N and including detunings) via Lax operators combined with a direct search by Menyuk *et al* (1983b), with different Lax operators by Wojciechowski *et al* (1986) and by a direct search based on what we have called irreducible forms (Verheest 1987b). These irreducible forms can be defined in different ways, perhaps easiest here as the simplest polynomials in a_j and $\bar{a}_j$ which are in involution with the Manley-Rowe relations (4), using the symmetries generated by these first integrals. For the set (1) the only irreducible forms are

$$\begin{aligned} &K_j = a_j\bar{a}_j \qquad j=0,\ldots,4 \\ &H_A = a_0\bar{a}_1\bar{a}_2 + \bar{a}_0 a_1 a_2 \qquad H_B = a_0\bar{a}_3\bar{a}_4 + \bar{a}_0 a_3 a_4 \\ &L = a_1 a_2 \bar{a}_3 \bar{a}_4 + \bar{a}_1\bar{a}_2 a_3 a_4. \end{aligned} \tag{5}$$

The idea then is to look for the remaining first integral as a polynomial in the irreducible forms, thus ensuring its involution with the Manley-Rowe relations.

The case $\mu = 2$ (or $\frac{1}{2}$), however, has so far proved intractable, apparently by whatever method. It is the purpose of the present letter to combine Yoshida's theorems (Yoshida 1983a, b), the results of the Painlevé analysis of Menyuk *et al* (1983a) and the concept of the irreducible forms (Verheest 1987b) to find a fifth independent first integral when $\mu = 2$ (or $\frac{1}{2}$). This follows the method proposed by Roekaerts and Schwarz (1987) except that the direct search is vastly simplified by the use of irreducible forms.

In order to facilitate the application of Yoshida's theorems and the computation of the Kovalevskaya exponents we formally rewrite (1) as a system of ODE in real variables a_j and A_j (e.g., by regarding $\tau = \mathrm{i}t$ as the new independent variable):

$$\begin{aligned} &\dot{a}_0 = a_1 a_2 + \mu a_3 a_4 \qquad \dot{A}_0 = -A_1 A_2 - \mu A_3 A_4 \\ &\dot{a}_1 = a_0 A_2 \qquad \dot{A}_1 = -A_0 a_2 \qquad \dot{a}_2 = a_0 A_1 \qquad \dot{A}_2 = -A_0 a_1 \\ &\dot{a}_3 = \mu a_0 A_4 \qquad \dot{A}_3 = -\mu A_0 a_4 \qquad \dot{a}_4 = \mu a_0 A_3 \qquad \dot{A}_4 = -\mu A_0 a_3. \end{aligned} \tag{6}$$

Here a_j and A_j are now real, standard canonically conjugate variables and (2) is replaced by

$$H = a_0 A_1 A_2 + A_0 a_1 a_2 + \mu(a_0 A_3 A_4 + A_0 a_3 a_4). \tag{7}$$

For the determination of the Kovalevskaya exponents one starts from the similarity transformation

$$t \to \alpha^{-1} t \qquad a_j \to \alpha^{g_j} a_j \qquad A_j \to \alpha^{G_j} A_j. \tag{8}$$

Applied to (6), this gives a set of possible g_j and G_j, which are here, in contrast to Yoshida's examples, not unique:

$$\begin{aligned} &g_0 = 3 - G_1 - G_2 \qquad G_0 = G_1 + G_2 - 1 \\ &g_1 = 2 - G_1 \qquad g_2 = 2 - G_2 \qquad g_3 = 2 - G_3 \\ &g_4 = 2 + G_3 - G_1 - G_2 \qquad G_4 = G_1 + G_2 - G_3. \end{aligned} \tag{9}$$

It is worth noting that for whatever choice of weights satisfying (9) the irreducible forms (5) are always weighted homogeneous of fixed degree (2, 3 or 4, respectively). Next one looks for a particular solution of (6) in the form

$$a_j = c_j(\tau - \tau_0)^{-g_j} \qquad A_j = C_j(\tau - \tau_0)^{-G_j} \tag{10}$$

and determines c_j and C_j. Because the weights g_j and G_j are not unique, many different possibilities arise, leading, as we shall see, to a certain ambiguity in the application of Yoshida's method. If $\mu \neq 1$, not all c_j, C_j can be chosen non-zero.

The Kovalevskaya exponents ρ are defined then as the roots of

$$K(\rho) \equiv \begin{vmatrix} \left.\dfrac{\partial f_i}{\partial a_j}\right|_c + (g_j - \rho)\delta_{ij} & \left.\dfrac{\partial f_i}{\partial A_j}\right|_c \\ \left.\dfrac{\partial F_i}{\partial a_j}\right|_c & \left.\dfrac{\partial F_i}{\partial A_j}\right|_c + (G_j - \rho)\delta_{ij} \end{vmatrix} = 0 \tag{11}$$

if the RHS of the equations (6) are denoted by f_j and F_j and the derivatives are evaluated for a particular choice of c_j and C_j.

A first and perhaps most obvious choice would be to take $G_1 = G_2 = G_3 = 1$, leading to every a_j and A_j scaling with α or in (10) going as $(\tau - \tau_0)^{-1}$. This requires *inter alia* that

$$\begin{aligned} &(1 + c_0 C_0)c_j = (1 + c_0 C_0)C_j = 0 && j = 1, 2 \\ &(1 + \mu^2 c_0 C_0)c_k = (1 + \mu^2 c_0 C_0)C_k = 0 && k = 3, 4. \end{aligned} \tag{12}$$

For the case at hand, where we want ultimately $\mu = 2$ (or $\frac{1}{2}$), we are forced to choose

$$\begin{aligned} &c_0 = -1/C_1 C_2 \qquad C_0 = C_1 C_2 \\ &c_1 = C_1^{-1} \qquad c_2 = C_2^{-1} \qquad c_3 = c_4 = C_3 = C_4 = 0 \end{aligned} \tag{13}$$

or something equivalent. Equation (11) then leads to

$$K(\rho) = (\rho + 1)(\rho - 3)\rho^2(\rho - 2)^2(\rho - \mu - 1)^2(\rho + \mu - 1)^2 = 0. \tag{14}$$

Yoshida's results indicate that if there exists a first integral ϕ of weighted degree M for a Hamiltonian system with Hamiltonian of weighted degree h, then M and $h - 1 - M$ appear as a pair of Kovalevskaya exponents, provided ϕ has a non-zero and finite gradient at the point $a_j = c_j$, $A_j = C_j$. As H is of degree 3, one is to find M and $2 - M$ as pairs. For $\mu = 2$ equation (14) gives a double pair $(2, 0)$ and a triple pair $(3, -1)$. The double pair $(2, 0)$ corresponds to the Manley–Rowe relations (except for E_B which has a vanishing gradient for the choice (13)). The triple pair $(3, -1)$ is not only indicative of H, but suggests additional first integrals of degree 3. Of these, however, there are none, as a direct search has shown. For $\mu = \frac{1}{2}$ one gets a double pair $(\frac{3}{2}, \frac{1}{2})$, indicating not a simple polynomial.

So one returns to (9) and then chooses $G_1 = G_2 = G_3 = 2$ and gets

$$\begin{aligned} &c_0 = 0 \qquad C_0 = 12[(1 - \mu^2)c_1 c_2]^{-1} \\ &c_1 C_1 = c_2 C_2 = -c_3 C_3 = 6(1 - \mu^2)^{-1} \\ &c_4 = -c_1 c_2(\mu c_3)^{-1} \qquad C_4 = 6\mu c_3[(1 - \mu^2)c_1 c_2]^{-1}. \end{aligned} \tag{15}$$

Now (11) yields

$$K(\rho) = (\rho + 1)(\rho - 3)\rho^3(\rho - 2)^3(\rho + 4)(\rho - 6) = 0 \tag{16}$$

independent of μ. This induces one to look for a polynomial of degree six as the missing first integral.

At this stage one returns to (1) and conducts a direct search. A fully general expression of degree 6 in the ten variables a_j and $\bar{a}_j$ would require *a priori* 5005 terms. Hence the great economy in the use of irreducible forms, which combine into a general polynomial of degree six with only 43 terms.

The missing first integral was thus duly found, after a long and tedious calculation. Armed with hindsight, it is perhaps (as usual!) better and simpler to give a more direct and constructive approach, in view of other applications and further extensions, which are now under study.

We see in (4) that the quadratic irreducible form K_0 can be eliminated via E_0. If we define

$$\mathcal{K}_A = K_1 + K_2 = a_1\bar{a}_1 + a_2\bar{a}_2 \qquad \mathcal{K}_B = K_3 + K_4 = a_3\bar{a}_3 + a_4\bar{a}_4 \tag{17}$$

then we can replace K_1, K_2 by $\mathcal{K}_A$, E_A and K_3, K_4 by $\mathcal{K}_B$, E_B. In view of the fact that

$$\dot{L} = \frac{1}{2\mu}\mathcal{K}_A\dot{\mathcal{K}}_B + \frac{\mu}{2}\mathcal{K}_B\dot{\mathcal{K}}_A \tag{18}$$

we attempt a general form for the missing first integral, presumed to be of degree six:

$$I = L(\alpha\mathcal{K}_A + \beta\mathcal{K}_B + \gamma E_A + \delta E_B + \varepsilon E_0) + \xi H_A^2 + \zeta H_B^2 + P \tag{19}$$

with P a homogeneous polynomial of degree three in $\mathcal{K}_A$, $\mathcal{K}_B$, E_A, E_B and E_0. No term in H_AH_B was written, because it can be eliminated through the square of the Hamiltonian $H_A + \mu H_B$. Using (18) and

$$\begin{aligned} L\dot{\mathcal{K}}_A &= \frac{2}{\mu}H_A\dot{H}_A + \frac{1}{2\mu}(\mathcal{K}_A^2 - E_A^2)\dot{\mathcal{K}}_B \\ L\dot{\mathcal{K}}_B &= 2\mu H_B\dot{H}_B + \frac{\mu}{2}(\mathcal{K}_B^2 - E_B^2)\dot{\mathcal{K}}_A \end{aligned} \tag{20}$$

we see that

$$\begin{aligned} \dot{I} = {} & \left(\frac{2\alpha}{\mu} + 2\xi\right)H_A\dot{H}_A + (2\mu\beta + 2\eta)H_B\dot{H}_B \\ & + \left(\frac{\alpha\mu}{2}\mathcal{K}_A\mathcal{K}_B + \beta\mu\mathcal{K}_B^2 - \frac{\beta\mu}{2}E_B^2 + \frac{\mu}{2}(\gamma E_A + \delta E_B + \varepsilon E_0)\mathcal{K}_B + \frac{\partial P}{\partial\mathcal{K}_A}\right)\dot{\mathcal{K}}_A \\ & + \left(\frac{\alpha}{\mu}\mathcal{K}_A^2 + \frac{\beta}{2\mu}\mathcal{K}_A\mathcal{K}_B - \frac{\alpha}{2\mu}E_A^2 + \frac{1}{2\mu}(\gamma E_A + \delta E_B + \varepsilon E_0)\mathcal{K}_A + \frac{\partial P}{\partial\mathcal{K}_B}\right)\dot{\mathcal{K}}_B. \end{aligned} \tag{21}$$

This can indeed be made zero by taking

$$\begin{aligned} &\xi = -\alpha/\mu \qquad \eta = -\beta\mu \\ &\frac{\partial P}{\partial\mathcal{K}_A} = -\frac{\alpha\mu}{2}\mathcal{K}_A\mathcal{K}_B - \beta\mu\mathcal{K}_B^2 + \frac{\beta\mu}{2}E_B^2 - \frac{\mu}{2}(\gamma E_A + \delta E_B + \varepsilon E_0)\mathcal{K}_B \\ &\frac{\partial P}{\partial\mathcal{K}_B} = -\frac{\alpha}{\mu}\mathcal{K}_A^2 - \frac{\beta}{2\mu}\mathcal{K}_A\mathcal{K}_B + \frac{\alpha}{2\mu}E_A^2 - \frac{1}{2\mu}(\gamma E_A + \delta E_B + \varepsilon E_0)\mathcal{K}_A \end{aligned} \tag{22}$$

with the integrability condition

$$\begin{aligned} &\frac{\alpha\mu}{2}\mathcal{K}_A + 2\beta\mu\mathcal{K}_B + \frac{\mu}{2}(\gamma E_A + \delta E_B + \varepsilon E_0) \\ &\qquad = \frac{2\alpha}{\mu}\mathcal{K}_A + \frac{\beta}{2\mu}\mathcal{K}_B + \frac{1}{2\mu}(\gamma E_A + \delta E_B + \varepsilon E_0). \end{aligned} \tag{23}$$

A detailed inspection of what $\dot{\mathscr{K}}_A$, $\dot{\mathscr{K}}_B$, $\dot{H}_A$ and $\dot{H}_B$ stand for reveals that the conditions (22) are not only sufficient but also necessary for I to be a first integral. Condition (23) then gives what was already known from the Painlevé analysis, namely that for integrability $\mu=2$ (forcing then $\beta=0$) or $\mu=\frac{1}{2}$ (giving $\alpha=0$). In any case where $\mu\neq 1$, $\gamma=\delta=\varepsilon=0$.

With $\mu=2$, it follows from (22) that

$$P=-\tfrac{1}{2}\alpha\mathscr{K}_A^2\mathscr{K}_B+\tfrac{1}{2}\alpha E_A^2\mathscr{K}_B \tag{24}$$

and the first integral is thus (taking $\alpha=4$ to avoid fractions):

$$\begin{aligned} I &= 4L\mathscr{K}_A-2H_A^2-2\mathscr{K}_A^2\mathscr{K}_B+E_A^2\mathscr{K}_B \\ &= 4(a_1a_2\bar{a}_3\bar{a}_4+\bar{a}_1\bar{a}_2a_3a_4)(a_1\bar{a}_1+a_2\bar{a}_2)-2(a_0\bar{a}_1\bar{a}_2+\bar{a}_0a_1a_2)^2 \\ &\quad -[(a_1\bar{a}_1+a_2\bar{a}_2)^2+4a_1\bar{a}_1a_2\bar{a}_2](a_3\bar{a}_3+a_4\bar{a}_4). \end{aligned}$$

One can easily check that it is independent of the other first integrals. Since it was constructed as a function of the irreducible forms, it is automatically in involution with E_A, E_B and E_0, and as an invariant of the motion also with H. This completes the direct proof of the integrability of (1) also for $\mu=2$.

The main lesson to be drawn from this letter, besides the desired proof of integrability, is that the irreducible forms are quite powerful in simplifying a direct search, especially when coupled to the Painlevé and Yoshida-Kovalevskaya analyses. It is also seen that the restrictions on μ follow in a natural way, should one not know them *a priori*. Work is in progress to extend the present results to the case of more interacting triplets and to also include detunings.

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